

Completeness Theorems for First-Order Logics of Formal Inconsistency

Rodrigo Podiacki and Walter Carnielli

Centre for Logic, Epistemology and the History of Science – CLE

and

Department of Philosophy – UNICAMP

P.O. Box 6133, 13083-970

Campinas, SP, Brazil

podiacki@gmail.com, carniell@cle.unicamp.br

Abstract

We investigate the question of characterizing first-order **LFIs** (Logics of Formal Inconsistency) by means of two-valued semantics. Although we focus on a particular study-case, the quantified logic **QmbC**, it will be shown how the method proposed here can be extended to a large family of quantified paraconsistent logics, supplying a sound and complete semantical interpretation for such logics.

1 First-order Logics of Formal Inconsistency

The *Logics of Formal Inconsistency* — from now on **LFIs** — are logics able to internalize, in a precise sense, the notions of “consistency” and “inconsistency” in the object-language level, be it by introducing primitive unary connectives, be it by appropriate definitions using the common propositional connectives. Such logics are paraconsistent in the following sense: given a contradiction of the form $(\varphi \wedge \neg\varphi)$, in general it is not possible to deduce an arbitrary formula ψ from it, that is, they do not fall into deductive triviality when exposed to a contradiction. This means that the Principle of Explosion, or Principle of Pseudo-Scotus, is not valid in general. However, **LFIs** may “explode” if φ , besides being contradictory, is also consistent (intuitively, this means that φ behaves classically, and cannot bear its own contradictory). So **LFIs** are submitted to a more restricted principle of explosion, called in [Carnielli, Coniglio and Marcos, 2007] the *Gentle Principle of Explosion*: an **LFI** explodes in the presence of φ , $\neg\varphi$ and $\circ\varphi$, for any arbitrary φ , where ‘ $\circ\varphi$ ’ expresses the fact that φ is consistent.

In its beginnings paraconsistent logic was only developed syntactically, that is, it was presented as an axiomatized calculus without any interpretation. In the

70s, there arose the first semantics, known as *valuation semantics*,¹ for the hierarchy of propositional paraconsistent logics C_n of da Costa. Nevertheless, the problem concerning convenient interpretation for first-order paraconsistent logic persisted. In 1984, Alves proposed a method which can be called *pre-structural semantics*. This method will be adopted in some extent in the present work and will be explained in what follows.

Definition 1. A first-order paraconsistent language L is composed by:

- a set of individual variables;
- for each $n \in \omega$, a set of function symbols of arity n ;
- for each $n \in \omega$, a set of predicate symbols of arity n ;
- connectives: $\neg, \circ, \wedge, \vee, \rightarrow$;
- quantifiers: \exists, \forall ;
- punctuation marks.

The only difference between paraconsistent and classical first-order languages is the presence, in the former, of the unary connective \circ , so that the expression ‘paraconsistent language’ is used just to keep the distinction clear. As usual, given a first-order language L , we assume it has at least one predicate symbol. Function symbols of arity 0 are individual constants, and predicate symbols of arity 0 are propositional constants. The definitions of *term*, *formula*,² and *atomic formula* are the usual ones, with the addition of the following clause: if φ is a formula, so it is $\circ\varphi$.

Definition 2. Let φ and ψ be formulas. If φ can be obtained from ψ by means of addition or deletion of void quantifiers, or by renaming of bound variables, we say that φ and ψ are variants of each other.³

The logic **mbC** was introduced in [Carnielli, Coniglio and Marcos, 2007] as a “fundamental” **LFI**, meaning that its axioms embody a minimum proof capability in order to preserve the positive theorems of classical propositional logic, and at the same time being capable of avoiding trivialization in the presence of contradictions. The axioms of **mbC** are 1 to 11 bellow. We set the following axioms to **QmbC**, which is the extension of **mbC** to first-order logic:

- 1) $\alpha \rightarrow (\beta \rightarrow \alpha)$
- 2) $(\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma))$
- 3) $\alpha \rightarrow (\beta \rightarrow (\alpha \wedge \beta))$
- 4) $(\alpha \wedge \beta) \rightarrow \alpha$

¹[da Costa and Alves 1977].

²We always put ‘formula’ for ‘well-formed formula.’

³See [Avron and Zamansky, 2006].

- 5) $(\alpha \wedge \beta) \rightarrow \beta$
- 6) $\alpha \rightarrow (\alpha \vee \beta)$
- 7) $\beta \rightarrow (\alpha \vee \beta)$
- 8) $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$
- 9) $\alpha \vee (\alpha \rightarrow \beta)$
- 10) $\alpha \vee \neg\alpha$
- 11) $\circ\alpha \rightarrow (\alpha \rightarrow (\neg\alpha \rightarrow \beta))$
- 12) $\varphi[x/t] \rightarrow \exists x\varphi$, if t is a term free for x in φ
- 13) $\forall x\varphi \rightarrow \varphi[x/t]$, if t is a term free for x in φ
- 14) If α is a variant of β , then $\alpha \rightarrow \beta$ is an axiom

Rules:

- *Modus ponens*;
- **Introduction of the universal quantifier:** $\alpha \rightarrow \beta / \alpha \rightarrow \forall x\beta$, if x isn't free in α ;
- **Introduction of the existential quantifier:** $\alpha \rightarrow \beta / \exists x\alpha \rightarrow \beta$, if x isn't free in β .

Some observations concerning notation. Given a first-order paraconsistent language L , the algebra of formulas determined by its signature is denoted by ' For_L° ', and ' S_L° ' indicates the set of closed formulas of L . ' $\mathcal{A}_{\mathcal{F}}(L)$ ' and ' $\mathcal{A}_{\mathcal{S}}(L)$ ' denote, respectively, the set of atomic formulas and the set of atomic sentences of L . If $\varphi[x_1, \dots, x_n]$ is a formula with n free variables, and t_1, \dots, t_n are terms of L , we put ' $\varphi[x_1/t_1, \dots, x_n/t_n]$ ' for the result of substituting each term t_i for the variable x_i , $1 \leq i \leq n$. The interpretation of a term t (formula φ) in a pre-structure \mathfrak{A} is denoted by ' $\mathfrak{A}(t)$ ' (' $\mathfrak{A}(\varphi)$ '). If ' f ' and ' P ' are, respectively, a function symbol and a predicate symbol of L , ' $f_{\mathfrak{A}}$ ' and ' $P_{\mathfrak{A}}$ ' denote, respectively, a function and a predicate in the (pre)structure \mathfrak{A} .

Definition 3. A pre-structure \mathfrak{A} for L is a pair $\langle |\mathfrak{A}|, I \rangle$ such that:

- $|\mathfrak{A}|$ is a non-empty domain of individuals;
- I is an interpretation function for each symbol of L , as usual. That is, for each n -ary function symbol f of L , I picks out a function $f_{\mathfrak{A}}: |\mathfrak{A}|^n \rightarrow |\mathfrak{A}|$. For each n -ary predicate symbol P of L , I selects an n -ary predicate $P_{\mathfrak{A}}$ in the domain, that is, a subset of $|\mathfrak{A}|^n$.

As it is clear, a pre-structure is exactly what a structure is in first-order classical semantics. The need for this difference in terminology will be explained soon.

Definition 4. Let $\mathfrak{A} = \langle |\mathfrak{A}|, I \rangle$ be a pre-structure for a language L . $L(\mathfrak{A})$ is the language obtained from L by addition, as new symbols, of the set of constants $\{\bar{i} : i \in |\mathfrak{A}|\}$. $\mathfrak{A}' = \langle |\mathfrak{A}|, I' \rangle$ is a pre-structure for $L(\mathfrak{A})$, such that I' is an extension of I satisfying: $\mathfrak{A}'(\bar{i}) = i$.

From now on a fixed paraconsistent first-order language L will be presupposed, as well as a pre-structure \mathfrak{A} for it. Hence, the extended language $L(\mathfrak{A})$ and the pre-structure \mathfrak{A}' for $L(\mathfrak{A})$ are determined from the fixed L and \mathfrak{A} . We shall be always working with \mathfrak{A}' and $L(\mathfrak{A})$. (Sometimes with further extensions.)

Definition 5. A *QmbC-valuation based on the pre-structure \mathfrak{A}'* is a function $v : S_{L(\mathfrak{A})}^\circ \longrightarrow \{0, 1\}$ satisfying:

1. $v(P(t_1, \dots, t_n)) = 1 \Leftrightarrow \langle \mathfrak{A}'(t_1), \dots, \mathfrak{A}'(t_n) \rangle \in P_{\mathfrak{A}'}$
2. $v(\alpha \vee \beta) = 1 \Leftrightarrow v(\alpha) = 1$ or $v(\beta) = 1$
3. $v(\alpha \wedge \beta) = 1 \Leftrightarrow v(\alpha) = 1$ and $v(\beta) = 1$
4. $v(\alpha \rightarrow \beta) = 1 \Leftrightarrow v(\alpha) = 0$ or $v(\beta) = 1$
5. $v(\alpha) = 0 \Rightarrow v(\neg\alpha) = 1$
6. $v(\circ\alpha) = 1 \Rightarrow v(\alpha) = 0$ or $v(\neg\alpha) = 0$
7. $v(\exists x\varphi) = 1 \Leftrightarrow v(\varphi[x/t]) = 1$ for some term t in $L(\mathfrak{A})$
8. $v(\forall x\varphi) = 1 \Leftrightarrow v(\varphi[x/t]) = 1$ for all t in $L(\mathfrak{A})$
9. If φ is a variant of ψ , then $v(\varphi) = v(\psi)$

In virtue of clauses 5 and 6 above, it is immediate to see that the following problem arises: given $L(\mathfrak{A})$ and \mathfrak{A}' , the mere assignment of values to all the atomic formulas of $L(\mathfrak{A})$, according to clause 1, will not be sufficient *per se* to determine the values of all its formulas in general. In other words, once $v(\varphi)$ is assigned for all $\varphi \in \mathcal{A}_S(L(\mathfrak{A}))$, there are more than one way of extending that atomic valuation to the set $S_{L(\mathfrak{A})}^\circ$. In fact, the next lemma shows that there are infinite ways of extending an atomic valuation. Hence, in order to obtain an adequate semantics for the calculus we are investigating, we need a stronger notion than the mere notion of pre-structure.

Definition 6. The complexity of a formula is a function $l : For_L^\circ \longrightarrow \mathbb{N}$ such that: $l(\varphi) = 0$, if $\varphi \in \mathcal{A}_F(L)$; $l(\psi\#\xi) = l(\psi) + l(\xi) + 1$, for $\# \in \{\vee, \wedge, \rightarrow\}$; $l(\neg\varphi) = l(\varphi) + 1$; $l(\circ\varphi) = l(\varphi) + 2$; and $l(Qx\varphi) = l(\varphi) + 1$, for $Q \in \{\forall, \exists\}$.

Lema 7. Let $v_0 : \mathcal{A}_S(L(\mathfrak{A})) \cup \{\neg\varphi : \varphi \in \mathcal{A}_S(L(\mathfrak{A}))\} \longrightarrow \{0, 1\}$ be a function satisfying clauses 1 and 5 of the preceding definition. Then there exists a *QmbC-valuation* $v : S_{L(\mathfrak{A})}^\circ \longrightarrow \{0, 1\}$ based on \mathfrak{A}' which extends v_0 , that is, such that $v(\varphi) = v_0(\varphi)$ for all formulas $\varphi \in \mathcal{A}_S(L(\mathfrak{A})) \cup \{\neg\varphi : \varphi \in \mathcal{A}_S(L(\mathfrak{A}))\}$.

Proof. Just let $v(\varphi) = v_0(\varphi)$ for all $\varphi \in \mathcal{A}_S(L(\mathfrak{A})) \cup \{\neg\varphi : \varphi \in \mathcal{A}_S(L(\mathfrak{A}))\}$, and put the values of formulas of greater complexity according to the clauses of Definition 5. \square

Definition 8. A **QmbC**-structure — or simply a structure, if no confusion can arise — for a first-order language L is a pair $\mathfrak{E} = \langle \mathfrak{A}, v \rangle$ such that \mathfrak{A} is a pre-structure for L , and v is a **QmbC**-valuation — or simply a valuation — based on \mathfrak{A} .

Given an structure \mathfrak{E} for L (based on \mathfrak{A}), we define a truth-value $\mathfrak{E}(\varphi)$ for each closed formula φ of $L(\mathfrak{A})$ in the following manner:

$$\mathfrak{E}(\varphi) = v(\varphi),$$

where v is the valuation of $\mathfrak{E} = \langle \mathfrak{A}, v \rangle$. If φ is a formula of L , an \mathfrak{A} -instance of φ is defined in a way similar of that of [Shoenfield, 1967], p. 19: it is a closed formula of the form $\varphi[x_1/t_1, \dots, x_n/t_n]$ in $L(\mathfrak{A})$, such that t_1, \dots, t_n are closed terms of $L(\mathfrak{A})$.

Finally, we can define ‘validity’ according to our semantics.

Definition 9. A formula φ of a language L is said to be:

1. valid in the structure \mathfrak{E} , if $\mathfrak{E}(\varphi) = 1$ for all \mathfrak{A} -instance φ' of φ ;
2. valid in the pre-structure \mathfrak{A} , if φ is valid in every structure \mathfrak{E} based on \mathfrak{A} ;
3. valid, if φ is valid in every structure \mathfrak{E} for L .

As usual, the fact that a formula α is valid in every structure in which all formulas of a set Γ are also valid is indicated by ‘ $\Gamma \models_{\mathbf{QmbC}} \alpha$.’ Given the above definition of ‘validity,’ (in **QmbC**) it is routine to verify that the proposed semantics is sound: just check that all the axioms of **QmbC** are valid and that the rules of inference preserve validity. Next we turn on the definitions which are necessary in order to demonstrate the completeness of our semantics.

Definition 10. Let $\Delta \subseteq For_{L(\mathfrak{A})}^\circ$ be a set, and \mathfrak{E} be a **QmbC**-structure for $L(\mathfrak{A})$ based on \mathfrak{A}' . We say that \mathfrak{E} is a model of Δ if and only if $\mathfrak{E}(\delta) = 1$ for every $\delta \in \Delta$. We indicate this fact by ‘ $\mathfrak{E} \models_{\mathbf{QmbC}} \Delta$.’

Definition 11. A set of formulas $\Delta \subseteq For_L^\circ$ is trivial in **QmbC** if and only if $\Delta \vdash_{\mathbf{QmbC}} \alpha$, for every $\alpha \in For_L^\circ$; otherwise Δ is non-trivial.

Definition 12. A set Δ whose formulas have as its language some extension L'^4 of L is a Henkin set if and only if (i) for all closed formula $\exists x\varphi \in \Delta$ there is a term t of $L'(\mathfrak{A})$ such that $\exists x\varphi \rightarrow \varphi[x/t] \in \Delta$, and (ii) if $\varphi[x/t] \in \Delta$ for every term t of L' free for x in φ , then $\forall x\varphi \in \Delta$.

We omit the proof of the following lema.

Lema 13. $(\alpha \rightarrow \beta) \rightarrow \perp \vdash_{\mathbf{QmbC}} \alpha \wedge (\beta \rightarrow \perp)$.

⁴Typically, an extension by addition of constants. In order to keep generality, we consider a language L as an extension of itself by adding the set \emptyset of constants

Theorem 14. *If $\Delta \subseteq \text{For}_L^\circ$ is a non-trivial set of formulas in \mathbf{QmbC} , then there is a set $\Gamma \supseteq \Delta$ of formulas of L' , for some extension L' of L , such that Γ is a Henkin set.*

Proof. We follow [Henkin, 1949]. Let Δ_0 be some set satisfying the conditions of the theorem. As the formulas of Δ_0 are written in some language L – which, by the way, will be denoted by $L(\Delta_0)$ –, we choose new constants u_κ^α (α, κ arbitrary ordinals) which do not belong to $L(\Delta_0)$. If the language of the set Δ_λ , to be constructed in the next paragraph, is already defined, the language of the set $\Delta_{\lambda+1}$ is built by the addition of the constants u_κ^α . If λ is a limit-ordinal and $L(\Delta_\sigma)$ if already defined for all $\sigma < \lambda$, then $L(\Delta_\lambda) = \bigcup_{\sigma < \lambda} L(\Delta_\sigma)$. For instance,

the language of the set Δ_1 is the one obtained from $L(\Delta_0)$ by the addition of the constants u_κ^1 , and $L(\Delta_\omega) = \bigcup_{n < \omega} L(\Delta_n)$. Let $L(\Delta_V)$ be the language which

contains all the constants u_κ^α of each $L(\Delta_\lambda)$, that is, $L(\Delta_V) = \bigcup L(\Delta_\lambda)$.

Next, we inductively build a chain Δ_λ of sets whose formulas have as their language $L(\Delta_\lambda)$. Let φ_κ^0 be the κ -th formula of Δ_0 of the form $\exists x\psi$. Then we put Δ_1 as the set obtained from Δ_0 by adding (i) the formula $\varphi_\kappa^0 \rightarrow \psi[x/u_\kappa^1]$, for each formula φ_κ^0 of the form $\exists x\psi$ of Δ_0 and all the constants u_κ^1 of $L(\Delta_1)$, and (ii) if $\varphi[x/t] \in \Delta_0$ for all terms t of $L(\Delta_0)$ free for x in φ , then we put $\forall x\varphi \in \Delta_1$. The formula $\exists x\psi \rightarrow \psi[x/u_\kappa^\alpha]$ is called the *special formula* for the constant u_κ^α . In general, if the set Δ_λ is already constructed, and if φ_κ^λ is the κ -th formula of Δ_λ of the form $\exists x\psi$, then $\Delta_{\lambda+1}$ is the set obtained from Δ_λ by the addition of (i) the formula $\varphi_\kappa^\lambda \rightarrow \psi[x/u_\kappa^{\lambda+1}]$, for each formula φ_κ^λ of the form $\exists x\psi$ of Δ_λ and all the constants $u_\kappa^{\lambda+1}$ of $L(\Delta_{\lambda+1})$, and (ii) if $\varphi[x/t] \in \Delta_\lambda$ for all terms t of $L(\Delta_\lambda)$ free for x in φ , then $\forall x\varphi \in \Delta_{\lambda+1}$. If λ is a limit-ordinal and Δ_σ is already constructed for all $\sigma < \lambda$, then $\Delta_\lambda = \bigcup_{\sigma < \lambda} \Delta_\sigma$. Finally, we postulate

$\Delta_V = \bigcup \Delta_\lambda$. By construction, Δ_V is a Henkin set such that $\Delta_0 \subseteq \Delta_V$.

It remains to be shown that Δ_V is a non-trivial set in \mathbf{QmbC} . If that was the case, then $\Delta_V \vdash_{\mathbf{QmbC}} \perp_{\Delta_0}$, for some bottom particle in $L(\Delta_0)$, hence also in $L(\Delta_\lambda)$, for all λ , because $\text{For}_{L(\Delta_0)}^\circ \subseteq \text{For}_{L(\Delta_\lambda)}^\circ \subseteq \text{For}_{L(\Delta_V)}^\circ$. As the proof of \perp_{Δ_0} from premisses in Δ_V involves just a finite number of formulas, and in particular a finite number of special formulas for the constants u_κ^α , it would be a proof of \perp_{Δ_0} from premisses in some Δ_λ . Hence it is sufficient to show that all Δ_λ are non-trivial, which is done by transfinite induction.

Suppose Δ_λ is non-trivial, but $\Delta_{\lambda+1} \vdash_{\mathbf{QmbC}} \perp_{\Delta_0}$. By construction, $\Delta_{\lambda+1} = \Delta_\lambda \cup \{\exists x\psi \rightarrow \psi[x/u_\kappa^{\lambda+1}]\}$. Then $\Delta_\lambda \cup \{\exists x\psi \rightarrow \psi[x/u_\kappa^{\lambda+1}]\} \vdash_{\mathbf{QmbC}} \perp_{\Delta_0}$. By the deduction theorem, $\Delta_\lambda \vdash_{\mathbf{QmbC}} (\exists x\psi \rightarrow \psi[x/u_\kappa^{\lambda+1}]) \rightarrow \perp_{\Delta_0}$. By the above lemma, $\Delta_\lambda \vdash_{\mathbf{QmbC}} \exists x\psi \wedge (\psi[x/u_\kappa^{\lambda+1}] \rightarrow \perp_{\Delta_0})$. Hence $\Delta_\lambda \vdash_{\mathbf{QmbC}} \psi[x/u_\kappa^{\lambda+1}] \rightarrow \perp_{\Delta_0}$. As the constant $u_\kappa^{\lambda+1}$ is new, it follows that $\Delta_\lambda \vdash_{\mathbf{QmbC}} \psi[x] \rightarrow \perp_{\Delta_0}$, and by existential introduction $\Delta_\lambda \vdash_{\mathbf{QmbC}} \exists x\psi \rightarrow \perp_{\Delta_0}$. But $\exists x\psi \in \Delta_\lambda$, hence $\Delta_\lambda \vdash_{\mathbf{QmbC}} \perp_{\Delta_0}$.

If λ is a limit-ordinal, suppose Δ_σ is non-trivial for all $\sigma < \lambda$. As $\Delta_\lambda =$

$\bigcup_{\sigma < \lambda} \Delta_\sigma$, if Δ_λ is trivial, then, for some σ , $\Delta_\sigma \vdash_{\mathbf{QmbC}} \perp_{\Delta_0}$, contradicting the hypothesis. \square

Definition 15. Let $\Gamma \cup \{\alpha\} \subseteq \text{For}_{L(\mathfrak{A})}^\circ$ be a set. Γ is said to be *maximal with respect to α* in \mathbf{QmbC} if $\Gamma \not\vdash_{\mathbf{QmbC}} \alpha$, and, for every formula $\beta \notin \Gamma$, it is the case that $\Gamma, \beta \vdash_{\mathbf{QmbC}} \alpha$. In which case Γ is also said to be *relatively maximal* (with respect to a given formula α).

Lema 16. Let $\Delta \cup \{\alpha\} \subseteq \text{For}_{L(\mathfrak{A})}^\circ$ such that $\Delta \not\vdash_{\mathbf{QmbC}} \alpha$. Then there is a set $\Gamma \supseteq \Delta$ of formulas in $L(\mathfrak{A})$ which is maximal with respect to α in \mathbf{QmbC} .

Proof. The proof is similar to that of Lindenbaum-Asser's lema. \square

Lema 17. Let L be a paraconsistent first-order language, and $\Delta \cup \{\alpha\}$ a set of formulas in For_L° such that Δ is maximal with respect to α in \mathbf{QmbC} . Then Δ is a closed theory.⁵

Proof. The proof of this lema, for \mathbf{mbC} , can be found in [Carnielli, Coniglio and Marcos 2007]. Its adaptation to \mathbf{QmbC} has no difficulties. Details are omitted in virtue of space. \square

Together, lema 16 and theorem 14 guarantee that every set of formulas in $L(\mathfrak{A})$, which is maximal with respect to some formula in \mathbf{QmbC} (and hence is non-trivial), can be extended to a Henkin set which is maximal with respect to the same formula. The language of formulas contained in this Henkin set, obtained from $L(\mathfrak{A})$ according to theorem 14, will be denoted by ' $L_H(\mathfrak{A})$ ', and the set of the formulas generated by this language will be designated by ' $\text{For}_{L_H(\mathfrak{A})}^\circ$ '. The following lema indicates some important properties of maximal Henkin sets in \mathbf{QmbC} .

Lema 18. Let $\Delta \subset \text{For}_{L_H(\mathfrak{A})}^\circ$ a Henkin set which is maximal with respect to a formula $\alpha \in \text{For}_{L_H(\mathfrak{A})}^\circ$ in \mathbf{QmbC} . Hence:

- (i) $(\beta \wedge \gamma) \in \Delta \Leftrightarrow \beta \in \Delta \text{ and } \gamma \in \Delta$
- (ii) $(\beta \vee \gamma) \in \Delta \Leftrightarrow \beta \in \Delta \text{ or } \gamma \in \Delta$
- (iii) $(\beta \rightarrow \gamma) \in \Delta \Leftrightarrow \beta \notin \Delta \text{ or } \gamma \in \Delta$
- (iv) $\beta \notin \Delta \Rightarrow \neg\beta \in \Delta$
- (v) $\circ\beta \in \Delta \Rightarrow \beta \notin \Delta \text{ or } \neg\beta \notin \Delta$
- (vi) $\exists x\varphi \in \Delta \Rightarrow \varphi[x/t] \in \Delta \text{ for some term } t \text{ in } L_H(\mathfrak{A})$
- (vii) $\forall x\varphi \in \Delta \Rightarrow \varphi[x/t] \in \Delta \text{ for every term } t \text{ in } L_H(\mathfrak{A})$
- (viii) If φ is a variant of ψ : $\varphi \in \Delta \Rightarrow \psi \in \Delta$

⁵If Γ is a set of formulas and Δ is the set of formulas which are consequences of Γ (in some logic), then Γ is a closed theory if and only if $\Gamma = \Delta$. See [Tarski, 1930], p. 33, where closed theories are called "deductive systems."

Proof. Item (i) is a consequence of the closure of Δ (previous lema), axioms 3, 4 and 5, and *modus ponens*. Item (ii) is a consequence of the closure of Δ , axioms 6, 7 and 8, and *modus ponens*. Item (iii) is a consequence of the closure of Δ , item (ii), axioms 1 and 9, and *modus ponens*. Item (iv) is a consequence of the closure of Δ , axiom 10, and *modus ponens*. For (v), suppose that $\beta \in \Delta$ and $\neg\beta \in \Delta$; then, for closure, relative maximality of Δ and axiom 11, it can be concluded that $\circ\beta \notin \Delta$. Items (vi) and (vii) are consequences of the closure of Δ , axioms 12 and 13, *modus ponens*, and the fact that Δ is a Henkin set. \square

Theorem 19. *Using the characteristic function of a Henkin set $\Delta \subset For_{L_H(\mathfrak{A})}^\circ$, which is relatively maximal in \mathbf{QmbC} , it is possible to define a pre-structure \mathfrak{B} for $L_H(\mathfrak{A})$ and \mathbf{QmbC} -valuation v_H based on \mathfrak{B} such that $v_H(\delta) = 1$ if and only if $\delta \in \Delta$.*

Proof. We employ the method of [Henkin, 1949]. A pre-structure \mathfrak{B} for $L_H(\mathfrak{A})$ and a \mathbf{QmbC} -valuation v_H based on \mathfrak{B} are constructed in the following manner. The domain $|\mathfrak{B}|$ of \mathfrak{B} is composed by the individual constants of $L_H(\mathfrak{A})$, which, by construction, has an infinite number of them. For each individual constant c of $L_H(\mathfrak{A})$, we define $\mathfrak{B}(c) = c$. For all n -ary function symbol f , $n \geq 1$, and all the closed terms t_1, \dots, t_n of $L_H(\mathfrak{A})$, we define $f_{\mathfrak{B}}(\mathfrak{B}(t_1), \dots, \mathfrak{B}(t_n)) = f(t_1, \dots, t_n)$. For each n -ary predicate symbol P of $L_H(\mathfrak{A})$, we define the predicate $P_{\mathfrak{B}}$ as the set of the n -tuples $\langle t_1, \dots, t_n \rangle$ such that $P(t_1, \dots, t_n) \in \Delta$. It is immediate, for the definition of each $P_{\mathfrak{B}}$ in \mathfrak{B} , that $\Delta \vdash_{\mathbf{QmbC}} P(t_1, \dots, t_n)$ if and only if $P(t_1, \dots, t_n) \in \Delta$ (since Δ is a closed theory), if and only if $\langle t_1, \dots, t_n \rangle \in P_{\mathfrak{B}}$ (for the definition of the predicate $P_{\mathfrak{B}}$). Since, by clause 1 of definition 5, $\langle t_1, \dots, t_n \rangle \in P_{\mathfrak{B}}$ if and only if $v(P(t_1, \dots, t_n)) = 1$, we conclude that, for all atomic formula $\varphi = P(t_1, \dots, t_n) \in \Delta$, $v_H(\varphi) = 1$.

It remains to define v_H for all the formulas of Δ . If φ has complexity n and $v_H(\varphi)$ is already defined, we put $v_H(\varphi') = 1$ if and only if $\varphi' \in \Delta$, for all φ' having complexity $n + 1$. Since Δ is a relatively maximal Henkin set, in virtue of lema 18, v_H satisfies all the clauses of definition 5. \square

Corolary 20. *The structure $\mathfrak{H} = \langle \mathfrak{B}, v_H \rangle$ is a model of Δ .*

Theorem 21. *Completeness. Let $\Gamma \cup \{\alpha\}$ be a set in $L(\mathfrak{A})$. Hence $\Gamma \models_{\mathbf{QmbC}} \alpha$ implica $\Gamma \vdash_{\mathbf{QmbC}} \alpha$.*

Proof. Let $\alpha \in For_{L(\mathfrak{A})}^\circ$ such that $\Gamma \not\vdash_{\mathbf{QmbC}} \alpha$. For theorem 14 and lema 16, it is possible to extend Γ to a Henkin set Γ^* which is relatively maximal with respect to α in \mathbf{QmbC} . As $\Gamma^* \not\vdash_{\mathbf{QmbC}} \alpha$, then $\alpha \notin \Gamma^*$. By theorem 19, it is possible to construct a pre-structure \mathfrak{B} and a \mathbf{QmbC} -valuation v_H based on \mathfrak{B} such that the structure $\mathfrak{H} = \langle \mathfrak{B}, v_H \rangle$ is a model of Γ^* , and $v_H(\varphi) = 1$ if and only if $\varphi \in \Gamma^*$. Hence, $\mathfrak{H} \models_{\mathbf{QmbC}} \Gamma^*$, and in particular $\mathfrak{H} \models_{\mathbf{QmbC}} \Gamma$; but $\mathfrak{H} \not\models_{\mathbf{QmbC}} \alpha$, which means that $\Gamma \not\vdash_{\mathbf{QmbC}} \alpha$. \square

2 Comments and perspectives

QmbC is a simple **LFI**. A natural way to reach stronger logics is by addition of axioms concerning e.g. double negations or “consistency propagation”. By means of combinations of such axioms, one obtains a rich variety of paraconsistent calculi having interesting properties: for example, the definition of some propositional connectives in terms of others.⁶ Consider the following schemata:

- (A) $(\circ\alpha \wedge \circ\beta) \rightarrow \circ(\alpha\#\beta)$
- (B) $(\circ\alpha \vee \circ\beta) \rightarrow \circ(\alpha\#\beta)$

$\# \in \{\vee, \wedge, \rightarrow\}$. If one adds axiom (A) or (B) to the remaining axioms of **QmbC**, he obtains two different kinds of “consistency propagation.” In case (A), in order that consistency can “propagate” from simple to more complex formulas, it is sufficient that both of its immediate subformulas are consistent; in case (B), it is enough that one of its subformulas be consistent. To see how the method developed here can be extended in order to obtain a complete semantics to the entire class of first-order **LFIs**, let **QmbC-A** the logic obtained by adding (A) to the axioms of **QmbC**. By adding the following clause to the definition 5

$$(A^*) \quad v(\circ\alpha \wedge \circ\beta) = 1 \Rightarrow v(\circ(\alpha\#\beta)) = 1$$

one obtains a **QmbC-A**-valuation. After a suitable definition of ‘structure,’ one reaches a **QmbC-A**-structure, and then proceeds as in the case of **QmbC** to establish a completeness theorem for this new calculus.

The possibility of obtaining a semantic characterization to the entire class of first-order **LFIs** is an interesting general result, since in this way we open up the way to a model-theoretical treatment to such logics. In principle, there is not much difficulty in variants of Compactness and Lowenheim-Skolem theorems for such logics, and from this ground many results in classical model theory can be regained.

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⁶See the logic labelled **Cio** in Carnielli, Coniglio and Marcos [2007].

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